Plural Quantifiers

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1 Expressive limitations of first-order logic

First-order logic uses only quantifiers that bind variables in name position. *Second-order* logic includes quantifiers that bind variables in predicate position. Thus, '$\exists X X a$' is a formula of second-order, but not first-order logic.

first-order logic
second-order logic

Sentences such as

(1) She is something I am not [namely, patient].

seem to be good candidates for second-order representation:

(2) $\exists X(Xs \wedge \neg Xi)$.

George Boolos [1] focuses on a different class of sentences, such as

(3) There are some critics who admire only one another.

(3) doesn't seem to be quantifying over properties, the way (1) is. It seem to be talking about the objects in the ordinary first-order domain. But still, somewhat surprisingly, it can't be captured in first-order terms.

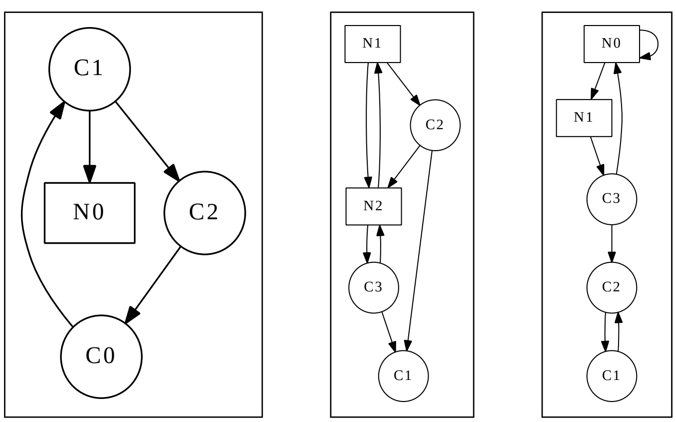

Figure 1: Which models make (3) true?

Let's think about what (3) says. It says that there are some critics, call them 'the Cs', such that none of the Cs admires herself or anyone outside of the Cs. To get clearer about it, try drawing some models in which it is true, and some in which it is false. Your models can consist of circles (critics), squares (non-critics), and arrows indicating who admires whom. (See for example Fig. 1.) In each model that makes (3) true, you should be able to answer the question: "Which are the critics who admire only one another?"

Try to convince yourself that (3) can't be formalized in first-order logic, by trying to formalize it. (The result can be proved—see below, §2.) If we are going to formalize (3), it seems, we must use second-order logic:

(4) $\exists X(\exists y Xy \wedge \forall y(Xy \supset Cy) \wedge \forall y \forall z[(Xy \wedge Ayz) \supset (y \neq z \wedge Xz)])$

Some of Boolos's examples have first-order equivalents, even though their most natural and direct representation is second-order. For example,

(5) There are some monuments in Italy of which no one tourist has seen all.

The natural second-order formalization is

(6) $\exists X(\exists x Xx \wedge \forall x(Xx \supset Mx) \wedge \neg\exists y(Ty \wedge \forall x(Xx \supset Syx)))$

But if you think about it, you can see that this is true just in case (a) there are some monuments in Italy, and (b) no tourist has seen all the monuments in Italy. And this, of course, can be expressed in first-order terms:

(7) $\exists x Mx \wedge \neg\exists x(Tx \wedge \forall y(My \supset Sxy))$.

My favorite example is one for which Quine thought there was a first-order equivalent, namely

(8) Some of Fiorecchio's men entered the building unaccompanied by anyone else.

As Boolos points out, the natural symbolization is second-order: 'anyone else' means 'anyone else but *them*.'

(9) $\exists X(\exists x Xx \wedge \forall x(Xx \supset Fx) \wedge \forall x(Xx \supset Ex) \wedge \forall x \forall y((Xx \wedge Axy) \supset Xy))$.

Quine analyzed this as

(10) $\exists x(Fx \wedge Ex \wedge \forall y(Axy \supset Fy))$.

Boolos argues that (10) can't be equivalent to (8), since (9) is provably non-first-orderizable.

> *Question:* Can you think of a situation in which (9) would be true but (10) false, under the intended interpretation (where 'Fx' means 'x is one of Fiorecchio's men', 'Ex' means 'x entered the building', etc.)? If not, how can we make sense of Boolos's claim? Is there a sense in which Quine is right? (Hint: Would things be any different if instead of 'unaccompanied by' we had 'unseen by'?)

So far we've been representing sentences using a second-order existential quantifier. We can also find cases that need a universal quantifier. Often these will combine 'some' and 'if.' Compare:

(11) a. If there is a horse who can beat Cody, ... it ...
$\forall x((Hx \wedge Bxc) \supset \ldots x \ldots)$

b. If there are some critics who admire only each other, …they …
$\forall X([\exists y Xy \wedge \forall y(Xy \supset Cy) \wedge \forall y \forall z((Xy \wedge Ayz) \supset (y \neq z \wedge Xz))] \supset \ldots X \ldots)$.

Here's an example:

(12) If there are some things of which 0 is one, and of which every successor of one of them is one, then every number is one of them.
$\forall X([X0 \wedge \forall x \forall y((Xx \wedge Syx) \supset Xy)] \supset \forall x(Nx \supset Xx))$

This, of course, is the principle of *mathematical induction*. It cannot be formalized in first-order terms. When we formalize arithmetic in first-order logic, we use instead an axiom *schema*, and posit as axioms all substitution instances of:

mathematical induction

(13) $[\phi 0 \wedge \forall x \forall y((\phi x \wedge Syx) \supset \phi y)] \supset \forall x(Nx \supset \phi x)$.

But this is much weaker than the second-order version; in effect, it quantifies only over groups of things that can be characterized with first-order formulas (the things that can be substituted for 'ϕ'). It's because of this that we get nonstandard models of arithmetic (on which see below, section 2).

Exercises:

1.1 Symbolize the following English sentences in second-order logical notation:

(a) Some boys and some girls went dancing, and each of the boys danced with at least one of the girls.

(b) There are some propositions such that the negation of any of them is one of them.

(c) Some critics admire only writers who hate most of them.

(d) If there are some people on the street, at least one of them will get shot.

(e) If some numbers are such that each of them is the product of two of them, then there are more than two of them.

1.2 For which of the above sentences can you find equivalent first-order formulas? (State the equivalent formulas, or 'can't find one.')

Extra credit:

1. Give an argument for the equivalence of (a) on p. 64 and (Z) on p. 63 of Boolos [1].

2. How would you symbolize 'Some teachers moved the piano across the room'?

2 Digression: proving unrepresentability

Skip this unless you're interested!

Boolos [1, p. 57] offers a proof (due to David Kaplan) that

(B) $\exists X(\exists x Xx \wedge \forall x \forall y[(Xx \wedge Axy) \supset (x \neq y \wedge Xy)])$

cannot be given a first-order formulation. The proof uses some concepts from metalogic, so don't worry if you can't understand it. For those who are interested, though, here's the basic idea:

1. If there were a first-order formula that captured the meaning of (B), it would be possible to give first-order axioms for arithmetic that rule out nonstandard models.
2. But it can be proven that no first-order axioms for arithmetic can rule out nonstandard models.
3. Hence (by reductio) there is no first-order formula that captures the meaning of (B).

What's a "nonstandard model" of arithmetic? Well, you know what a *model* is: a domain and an interpretation of the language's predicates and individual constants on that domain. A model of a set of axioms is a model that makes these axioms true. Now consider a set of first-order axioms for arithmetic (such as the standard Peano axioms). These axioms will contain some arithmetical expressions, like '0', 'S', '$+$', and '$<$'. The *standard model* of arithmetic interprets these in the obvious way: the domain is the set of natural numbers, the extension of 'S' is the set of pairs consisting of a natural number and its successor, the extension of '$+$' is the set of triples consisting of two natural numbers and their sum, and the extension of '$<$' is the set of pairs consisting of two natural numbers where the first is less than the second.

nonstandard models of arithmetic

But, surprisingly, the standard model is not the only model of the axioms. There are *nonstandard models* whose domains contain lots of "extra numbers" that are greater than all the standard natural numbers. These nonstandard numbers are numbers that you could never get to by starting with 0 and moving in a finite number of steps to the next number.

Proof of the existence of nonstandard models: The *compactness theorem* for first-order logic says that a set of sentences has a model iff every finite subset of the set has a model. Let A be your first-order axioms for arithmetic, and consider the set $S = A \cup \{Na, a \neq 0, a \neq 1, a \neq 2, \cdots\}$. Clearly, any finite subset T of this set has a model—just interpret a as the smallest natural number not mentioned in T. So, by compactness, S has a model. In this model, Na is true but a cannot denote any of the standard numbers.

You might think that the principle of mathematical induction rules out such numbers. In its natural second-order formulation (12), the principle says that any property that belongs to 0 and belongs to the successor of a number if it belongs to that number, belongs to all natural numbers. How could that be true if there are nonstandard numbers that can never be reached by starting with 0 and moving to the successor? But remember, in first order logic we just have an induction *schema*, (13). This ensures that any property that is *expressible in the language*, belongs to 0, and belongs to the successor of a number if it belongs to that number, belongs to all natural numbers. This might be true even if there are properties, inexpressible in the language, that belong to the standard natural numbers but not the nonstandard ones.

Kaplan establishes premise (1) of his argument by giving a sentence (C) that is a substitution instance of (B), with '$(x = 0 \vee x = y + 1)$' put in for 'Axy'. Clearly, if there is a first-order representation of (B), there will be a first-order representation of (C). He then shows that (C) is true in every nonstandard model of arithmetic, but false in the standard model. To see this, define 'x doodles y' as '$x = 0 \vee x = y + 1$', so that 0 doodles everything and other numbers doodle their predecessors. Then (C) says:

(14) There are some numbers that only doodle each other.

This will be false in standard models of arithmetic. Consider any group of standard numbers. If it contains 0, then it can't be a group of numbers that only doodle each other, since 0 doodles itself. If it doesn't contain 0, then let k be the least number it contains. Since k is the least number in the group, $k-1$ is not in the group. But k doodles $k-1$. So again, it can't be a group of numbers that only doodle each other.

But (14) is true in all nonstandard models of arithmetic, since these models contain infinite descending chains of numbers that don't bottom out in 0.

Thus, if there were a first-order formula equivalent to (C), we would have a first-order way to rule out all nonstandard models of arithmetic: just add the negation of (C) to the other axioms. Since it can be proven on general grounds that there is no way to do this, we know there can't be a first-order formula equivalent to (C).

3 Set Theory in Sheep's Clothing?

So, we've managed to formalize some sentences of English using second-order quantifiers, and we've seen that in many cases that is the *only* way we can formalize them. Let's now turn to the question of how, exactly, these second-order quantifiers are to be understood. For simplicity, let's focus on the simplest second-order sentence:

(15) $\exists X X a$

We might understand (15) to be saying one of the following:

(16) a. There is a property that a possesses.
b. There is a concept that applies to a.

c. There is a set of which a is a member.

All of these construals treat the second-order quantifier as quantifying over a special kind of entity (properties, concepts, sets). But then it is unclear why we could not simply include these entities in the domain of our first-order quantifiers, and use first-order logic:

(17) a. $\exists x Hax$

b. $\exists x Axa$

c. $\exists x a \in x$

For just this reason, Quine [3] famously objected that second-order logic was just "set theory in sheep's clothing." His point was that if we're going to be talking about sets, it's more honest just to make that explicit. (For a useful critical examination of Quine's charge, see [2].)

But there are reasons to be dissatisfied with this way of thinking about our second-order formalizations. For one thing, when we say

(18) There are some critics who admire only each other,

it doesn't seem as if we're talking about *sets* (or properties or concepts). We're just talking about critics. It seems perfectly coherent to say, for example,

(19) There are some critics who admire only each other, and there are no sets.

As Boolos says, "...there may be a set containing all trucks, but that there is certainly doesn't seem to *follow* from the truth of 'There are some trucks of which every truck is one.' " Indeed, this last sentence seems to say little more than that there are trucks.

Moreover, as Boolos points out, there are some things we'd *like* to be able to say about sets using our second-order language that we couldn't coherently say if our second-order quantifiers were really quantifiers over sets. For example (taking the first-order domain to be sets),

(20) $\exists X(\exists x Xx \wedge \forall x(x \notin x \supset Xx))$

There are some sets of which every set that is not a member of itself is one.

Boolos thinks that this is equivalent to the first-order sentence

(21) There are some sets and every set that is not a member of itself is a set,

which is true. But if we take (20) to be a disguised set-theoretic statement, namely

(22) There is a set of sets of which every set that is not a member of itself is a member,

we get something that is *false* in standard (Zermelo-Fraenkel or "ZF") set theory (for reasons relating to Russell's Paradox). (If there is such a set, is it a member of itself, or not?)

4 Boolos's plural interpretation

Boolos's alternative suggestion is that the second-order quantifiers don't range over anything other than the objects the first-order quantifiers range over them. It's just that they range over them *plurally* instead of singly. His punch line:

> The lesson to be drawn from the foregoing reflections on plurals and second-order logic is that neither the use of plurals nor the employment of second-order logic commits us to the existence of extra items beyond those to which we are already committed. We need not construe second-order quantifiers as ranging over anything other than the objects over which our first-order quantifiers range, and in the absence of other reasons for thinking so, we need not think that there are collections of (say) Cheerios, in addition to the Cheerios. Ontological commitment is carried by our *first*-order quantifiers; a second-order quantifier needn't be taken to be a kind of first-order quantifier in disguise, having items of a special kind, collections, in its range. (p. 72)

To support this claim, Boolos provides a procedure for translating any sentence of (monadic) second-order logic into a sentence of English that does not make any mention of sets or collections, but instead uses normal plural constructions that we understand independently of set theory. The English sentences that his translations produce are stilted and slightly augmented by subscripts (though these could be eliminated in principle by phrases such as 'the former' and 'the latter'). But they are "proper sentences of English which, with a modicum of difficulty, can be understood and seen to say something true."

The procedure is summarized in Fig. 2.

ϕ	$Tr(\phi)$
a	Alex [or whatever!]
Fv	it$_v$ is a frog [or whatever!]
Vv	it$_v$ is one of them$_V$
$v = v'$	it$_v$ is identical with it$_{v'}$
$\wedge$	and
$\vee$	or
$\neg$	it is not the case that
$\supset$	only if
$\equiv$	if and only if
$\exists v \psi$	there is something$_v$ such that $Tr(\psi)$
$\forall v \psi$	everything$_v$ is such that $Tr(\psi)$
$\exists V \psi$	either there are some things that$_V$ are such that $Tr(\psi)$, or $\psi^\perp$ [where $\psi^\perp$ is the result of substituting every occurrence of $\ulcorner V\alpha \urcorner$ in ψ, where α is any term, with $\ulcorner \alpha \neq \alpha \urcorner$]

Figure 2: Boolos's translation scheme.

No translation is given for universal second-order quantifiers; these must be handled using the equivalence $\forall X \Leftrightarrow \neg\exists X \neg$. The reason for the complicated clause for '$\exists V$' is that English "some things" implies "at least one thing," but the second-order quantifier does not. So, for example, '$\exists X \forall x(Xx \equiv x \neq x)$' is true, but 'there are some things such that a thing is one of them if and only if it is not identical to itself' is not. $\psi^{\perp}$ says, essentially, that ψ is true if nothing is V.

Let's try using this procedure to get a translation of

(23) $\exists X \forall x(Fx \equiv Xx)$.

The main operator is '$\exists X$', so we apply the last rule:

(24) Either there are some things that$_X$ are such that Tr('$\forall x(Fx \equiv Xx)$'), or Tr('$\forall x(Fx \equiv x \neq x)$').

The last part (after 'or') is just standard first-order stuff. So we just need to compute Tr('$\forall x(Fx \equiv Xx)$'). Applying the rule for '$\forall v$', we get:

(25) everything$_x$ is such that Tr('$Fx \equiv Xx$').

Tr('$Fx \equiv Xx$') is

(26) Tr('Fx') if and only if Tr('Xx').

Tr('Fx') is just 'it$_x$ is a frog,' and Tr('Xx') is 'it$_x$ is one of them$_X$.' Putting it all together then, we get:

(27) Either there are some things that$_X$ are such that everything$_x$ is such that it$_x$ is a frog if and only if it$_x$ is one of them$_X$, or everything is such that it$_x$ is a frog if and only if it$_x$ is not identical with itself.

Which is intelligible English, though not something you'd want to write! Once you have this, you can try to get a smoother version:

(28) Either there are some things such that a thing is one of them just in case it's a frog, or nothing is a frog.

Exercises: Translate the following second-order formulas into English, using Boolos's translation method. (You may use subscripted pronouns, like 'them$_X$,' but after giving a translation using these, try to give a more idiomatic version.) Take 'Fx' to mean 'x is a frog,' 'a' to mean 'Al' and 'b' to mean 'Bo'.

4.1 $\exists X(Xa \vee Xb)$

4.2 $\exists X(\forall x(Xx \supset Fx) \wedge \neg Xa)$

4.3 $\neg \forall X \forall x(Xx \supset Fx)$

References

[1] George Boolos. "To Be is to Be a Value of a Variable (or to Be Some Values of Some Variables)". In: *Journal of Philosophy* 81.8 (1984), pp. 430–449.

[2] George S. Boolos. "On Second-Order Logic". In: *Journal of Philosophy* 72 (1975), pp. 509–527.

[3] W. V. O. Quine. *Philosophy of Logic*. Cambridge, MA: Harvard University Press, 1970.